QUANTUM ERGODICITY FOR A CLASS OF MIXED SYSTEMS

JEFFREY GALKOWSKI

Abstract. We consider sets of quasimodes for the Dirichlet Laplacian on a domain with boundary where the geodesic flow exhibits mixed dynamical behavior. We assume that the billiard flow has an invariant ergodic component, U, and study defect measures, μ, of positive density sequences of almost orthogonal quasimodes. We demonstrate that these measures have $\mu|_U = c\mu_L|_U$ where μ_L is the Liouville measure. In order to do this, we adapt quantum ergodicity results to the case of quasimodes. Finally, using Bunimovich's mushroom billiards [2], we give an example where our results apply.

1. Introduction

The distribution of eigenfunctions over phase space in the semiclassical limit is an important object of study in the theory of quantum chaos. The fundamental result is a quantum ergodicity theorem of Shnirelman [14], Zelditch [15], and Colin de Verdière [4] which states that, for classically ergodic systems, high energy eigenfunctions distribute uniformly in phase space. Since we will study domains with boundary, we note that Zelditch-Zworski [16] generalized quantum ergodicity to that case.

Some progress has been made toward understanding semiclassical limits of eigenfunctions for systems with dynamical behavior that is not completely ergodic. Marklof and O'Keefe, [8], examine separated phase spaces for certain maps. More recently, Marklof and Rudnick [9] have made further strides towards an understanding of quantum behavior for mixed phase space. In particular, they prove that, for rational polygons, the eigenfunctions of the Dirichlet Laplacian equidistribute in configuration space. We use [9] as inspiration for further work on semiclassical limits for systems with mixed dynamical behavior.

We examine systems whose phase spaces have invariant subsets, U, such that the flow restricted to U is ergodic and the Liouville measure of U is positive. In particular, let (M, g) be a smooth Riemannian manifold of dimension d with a piecewise smooth boundary, ∂M. Then $M \subset \widetilde{M}$, where $\widetilde{M}$ is a manifold without boundary to which g extends smoothly, and $\partial M = \bigcup_{j=1}^{J} N_j$ where N_j are smooth embedded hypersurfaces in $\widetilde{M}$. Define $\partial^o M \subset \partial M$ to be the open set of all points where the boundary is smooth. Then the complement $\partial M \setminus \partial^o M$ has measure zero.

We consider $P(h) = -h^2\Delta + V$ with Dirichlet boundary conditions and let $p(x, \xi)$ be its principal symbol. Since $p(x, \xi)$ is smooth up to the boundary, we can extend it smoothly to $T^*\widetilde{M}$. We assume that

$$\text{for } E \in [a, b],\ dp|_{p^{-1}(E)} \neq 0 \tag{1.1}$$

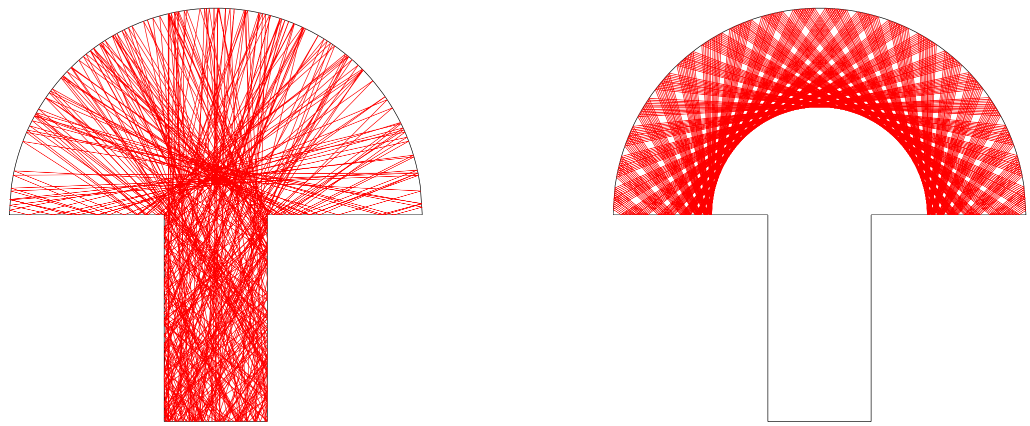

FIGURE 1. We show two billiards trajectories in a Bunimovich mushroom [2] with semicircular hat of radius 1 and centered base of width 1/2 and height 1. On the left, we have a trajectory in the ergodic portion of phase space. On the right, we have one in the integrable portion.

and

$$x \in \partial^o M, V(x) = E \Rightarrow dV \notin N^*_x \partial^o M. \tag{1.2}$$

so that $p^{-1}(E)$ and $T^*_{\partial^o M}M$ intersect transversally.

We then write

$$p^{-1}(E) \cap T^*_{\partial^o M}M = \Omega^+_E \sqcup \Omega^-_E \sqcup \Omega^0_E,$$

where (x, ξ) lies in Ω^+_E if the vector $H_p x \in T\widetilde{M}$ points outside of M, in Ω^-_E if it points inside M, and Ω^0_E if it is tangent to ∂M. The set Ω^0_E contains the glancing covectors and, under (1.2), has measure zero in $p^{-1}(E) \cap T^*_{\partial^o M}M$.

We define the broken Hamiltonian flow as follows. (see, for example, [5, Appendix A]). We denote this flow by $\varphi_t := \exp(H_p t)$. Assume without loss of generality that $t > 0$. We consider $\exp(tH_p)(x, \xi)$ defined on $T^*\widetilde{M}$, and let t_0 be the first nonnegative time when $\exp(tH_p)(x, \xi)$ hits the boundary. If this happens at a non-smooth point of the boundary, or if $\exp(t_0 H_p)(x, \xi) \in \Omega^0_E$, then the flow cannot be extended past $t = t_0$. Otherwise, $\exp(t_0 H_p)(x, \xi) \in \Omega^+_E$ and there exists unique $(x_0, \xi_0) \in \Omega^-_E$ such that the natural projections of $\exp(t_0 H_p)(x, \xi)$ and (x_0, ξ_0) onto $T^*\partial M$ are the same. We then define φ_t inductively, by putting $\varphi_t(x, \xi) = \exp(tH_p)(x, \xi)$ for $0 < t < t_0$ and $\varphi_t(x, \xi) = \varphi_{t-t_0}(x_0, \xi_0)$ for $t > t_0$. Define for $T > 0$, the set

$$\mathcal{B}_T \subset T^*M \cap p^{-1}([a, b])$$

to be the closed set of all (x, ξ) such that $\varphi_t(x, \xi)$ intersects a glancing point for some $t \in [-T, T]$. Then, as shown in [16, Lemma 1], $\mathcal{B}_T$ has measure zero in $p^{-1}(E)$.

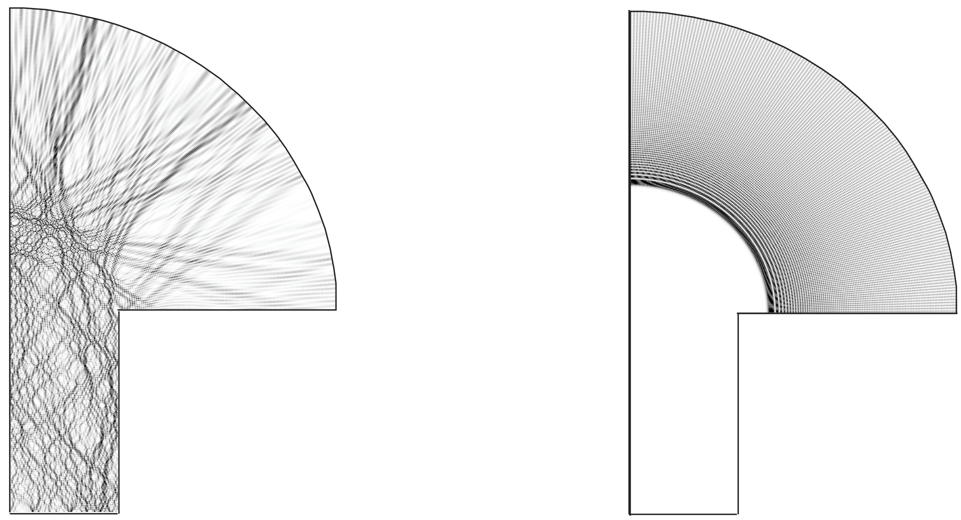

FIGURE 2. We show two high energy eigenfunctions for the laplacian in a Bunimovich mushroom [2]. The left hand eigenfunction spreads uniformly through the ergodic protion of phase space, and the right hand eigenfunction concentrates in the integrable portion. (Images courtesy of A. Barnett and T. Betcke [1]).

The ease with which quasimodes can be constructed in examples motivates us to generalize existing quantum ergodicity results to that setting. Although this is a natural generalization of quantum ergodicity results, we are not aware of any reference in the literature. We present this generalization in section 2. In order to do this, we make the following definition (with the obvious analog in the classical setting under the rescaling $h = \lambda^{-1}$).

Definition 1. *A positive density set of quasimodes for P on $[a, b]$, $0 \le a < b$ is a collection $\{(\psi_j, E_j), j = 1, \ldots\}$ satisfying,*

(1) $\|\psi_j\|_{L^2} = 1$
(2) $\|(P - E_j)\psi_j\|_{L^2} = o(h^{2d+1})$
(3) $|\langle \psi_j, \psi_k \rangle| = o(h^{2d}) \quad j \neq k$
(4) $|\{E_j \in [a, b]\}| \ge ch^{-d}$, $c > 0$.

Remark: Notice that the set of all eigenfunctions is a positive density set of quasimodes. Hence, all of our results will apply in this case.

Under assumptions (1.1), (1.2), and

$$\varphi_t \text{ is ergodic on } p^{-1}(E), \quad E \in [a, b] \tag{1.3}$$

we have the following analog of quantum ergodicity for a positive density set of quasimodes for P. For a semiclassical pseudodiffential operator of order m, we write $B \in \Psi_h^m$, and denote its semiclassical symbol, $\sigma_h(B)$ (see, for example, [17, Section 14.2] for definitions).

Theorem 1. *Suppose that (M, g) is a compact manifold with a piecewise smooth boundary and (1.3) holds. Let $\{(\psi_j, E_j), j = 1, ...\}$ be a positive density set of quasimodes of the Dirichlet realization of P on $[a, b]$. Then there exist a family of subsets*

$$\Lambda(h) \subset \{E_j | a \leq E_j \leq b\}$$

of full density such that for all $B \in \Psi_h^0(M^o)$ with symbol, $\sigma_h(B)$, compactly supported away from ∂M,

$$\langle B\psi_j, \psi_j\rangle \to \int_{p^{-1}(E_j)} \sigma_h(B) d\mu_{E_j}$$

for $E_j \in \Lambda(h)$.

In section 3, we specialize to the case $P = -h^2\Delta$ and pass from semiclassical to the classical setting. For a classical pseudodifferential operator of order m, we write $B \in \Psi^m$ and denote its classical symbol, $\sigma(B)$ (see, for example, [6, Chapter 18] for definitions).

Theorem 2. *Suppose that (M, g) is a compact manifold with a piecewise smooth boundary and (1.3) holds. Let $\{(\psi_j, E_j), j = 1, ...\}$ be a positive density set of quasimodes of the Dirichlet realization of $-\Delta$ on $[0, 1]$. Then, there is a full density subsequence ψ_{j_k} such that for all classical pseudodifferential operators $B \in \Psi^0(M^o)$ with symbol compactly supported away from ∂M,*

$$\langle B\psi_{j_k}, \psi_{j_k}\rangle \to \int_{S^*M} \sigma(B) d\mu_L.$$

In section 4, we continue to restrict our attention to the classical setting. But, instead of the ergodic assumption (1.3), we make the mixed dynamical hypothesis

$$\exists U \subset S^*M, \quad \mu_L(U) > 0, \quad \mu_L(\partial U \setminus U) = 0 \tag{1.4}$$

$$\varphi_t \text{ is ergodic on } U, \quad \varphi_t(U) = U. \tag{1.5}$$

In this case, we have the following result

Theorem 3. *Suppose that (M, g) is a compact manifold with a piecewise smooth boundary and that (1.4), (1.5) hold. Let $\{(\psi_j, E_j), j = 1, ...\}$ be a positive density set of quasimodes of the Dirichlet realization of $-\Delta$ on $[0, 1]$. Then there is a full density subsequence of ψ_j such that if any further subsequence ψ_{j_k} has a defect measure, μ, $\mu|_U \equiv c\mu_L|_U$.*

Here, a defect measure for a sequence ψ_j is the limit of the semiclassical measures μ_j associated to ψ_j. (see, for example, [17, Chapter 5] for details) In the boundaryless case, Riviére [12] recently proved a theorem on accumulation points of semiclassical measures which gives similar results for separated phase spaces. He also provides examples of separated phase spaces for the geodesic flow on boundaryless manifolds.

Remark: Hypotheses (1.4) include the case where U is closed with positive measure boundary.

Finally, in section 5, we apply our results to the Dirichlet Laplacian for mushroom billiards. Figure 1 shows two billiards trajectories, one in the ergodic region and one in the integrable region. Percival [13] made conjectures on eigenfunctions that were numerically verified by Barnett and Betcke in [1] in the case of mushroom billiards. Our results provide rigorous proofs for some parts of these conjectures. (For a complete description of the billiard map on mushrooms see [2].)

Acknowledgemnts. The author would like to thank Maciej Zworski for valuable discussion, Stéphane Nonnenmacher and Zeév Rudnick for pointing him toward mushroom billiards and [9], to Semyon Dyatlov for advice on boundary value problem quantum ergodicity, and to Alex Barnett for allowing him to use the images in Figure 2. The author is grateful to the Erwin Schrödinger Institute for support during the Summer School on Quantum Chaos and to the National Science Foundation for support under the National Science Foundation Graduate Research Fellowship Grant No. DGE 1106400, and for support during the Erwin Schrödinger Institute's Summer School on Quantum Chaos.

2. Quantum Ergodicity for Quasimodes

In this section, we demonstrate how to adapt quantum ergodicity results for eigenfunctions to positive density sets of quasimodes. We make the following definition.

Definition 2. *Let E_j, $j = 1, ...$ be all the eigenvalues of P with multiplicity. A complete set of quasimodes for P is a collection $\{\psi_j, j = 1, ...\}$ satisfying*

$$||\psi_j||_{L^2} = 1, \quad ||(P - E_j)\psi_j||_{L^2} = o(h^{2d+1}), \quad |\langle \psi_j, \psi_k \rangle| = o(h^{2d}),\ j \neq k.$$

We will need the following lemmas relating various spectral quantities for operators to their corresponding expressions involving complete sets of quasimodes.

Lemma 1. *Let $\{u_k, k = 1, ...\}$ be the eigenfunctions of P corresponding to E_k and let $\{\psi_j, j = 1, ...\}$ be a complete set of quasimodes for P. Then,*

$$u_j = \sum_{|E_j - E_k| < Ch^{d+1}} c_{jk}\psi_k + o(h^d).$$

Proof. First, observe that, near an energy level E, there exists $(c_1(h), c_2(h))$ with $|c_1(h) - c_2(h)| \geq ch^{d+1}$ such that $((E - c_2(h), E - c_1(h)) \cup (E + c_1(h), E + c_2(h))) \cap \{E_j, j = 1, ...\} = \emptyset$ – if this were false, then the spectrum of P would violate the Weyl law.

Now, let $\Lambda = \{\psi_j | E_j \in (E - c_1(h), E + c_1(h))\}$ and Π be the spectral projection onto $[E - c_1(h), E + c_1(h)]$. By the Weyl law, $|\Lambda| \leq Ch^{-d}$. Therefore, by [7, Proposition 32.4],

$$\dim\ (\Pi \text{ span } (\Lambda)) = |\Lambda|,$$

since

$$o(h^{2d}) + o(h^{2d+1})O(h^{-d-1}) = o(h^d).$$

But, rank $\Pi = |\Lambda|$. Hence, span Λ = range Π and the result follows from the almost orthogonality of ψ_j. $\square$

Lemma 2. *Let A be a Hilbert-Schmidt operator and $\{\psi_j, j = 1, ...\}$ be a complete set of quasimodes for P and $a < b$. Then,*

$$h^d \sum_{a \le E_j \le b} \|A\psi_j\|^2 = h^d \|\Pi_{[a,b]} A\|_{HS}^2 + o(1) \tag{2.1}$$

and if A is of trace class,

$$h^d \sum_{a \le E_j \le b} \langle A\psi_j, \psi_j\rangle = h^d Tr \Pi_{[a,b]} A + o(1). \tag{2.2}$$

Proof. First, let $(c_{jk}) = \langle u_k, \psi_j\rangle$. Then, by Lemma 1,

$$u_j = \sum_k c_{jk}\psi_k + o(h^d) \quad \psi_j = \sum_k \overline{c_{jk}} u_k,$$

where all sums are taken over k such that $|E_k - E_j| < Ch^{d+1}$. Observe that

$$u_j = \sum_k c_{jk}\psi_k + o(h^d) = \sum_{k,k'} c_{jk}\overline{c_{kk'}} u_{k'} + o(h^d).$$

Hence, by the orthogonality of u_j,

$$\sum_k c_{jk}\overline{c_{kk'}} = \delta_{kj} + o(h^d).$$

Now,

$$\begin{aligned} h^d \sum_{a \le E_j \le b} \langle A\psi_j, A\psi_j\rangle &= h^d \sum_{j,k,k'} c_{jk}\overline{c_{jk'}} \langle Au_k, Au_k'\rangle = h^d \sum_{k,k'} (\delta_{kk'} + o(h^d))\langle Au_k, Au_{k'}\rangle \\ &= h^d \|\Pi_{[a,b]} A\|_{HS}^2 + h^d \sum_{k,k'} o(h^d)\langle Au_k, Au_{k'}\rangle = h^d \|\Pi_{[a,b]} A\|_{HS}^2 + o(1) \end{aligned}$$

where the last equality follows from the fact that there are at most $O(h^{-d})$ terms in each sum. An analgous argument shows that (2.2) holds. □

To prove Theorem 1, we need the following lemma similar to [5, Lemma A.2]

Lemma 3. *Let $\chi \in C_c^\infty(M^o)$ with $0 \le \chi \le 1$. Then for $a' < a < b < b'$,*

$$(2\pi h)^d \sum_{E_j \in [a,b]} \int_M (1-\chi)|\psi_j|^2 dVol \le \int_{T^*M \cap p^{-1}([a',b'])} (1-\chi) d\mu_\sigma + o(1) \text{ as } h \to 0.$$

Proof. This follows directly from [5, Lemma A.2] and Lemma 2 once we observe that

$$(2\pi h)^d \sum_{E_j \in [a,b]} \int_M (1-\chi)|\psi_j|^2 dVol = (2\pi h)^d \sum_{E_j \in [a,b]} \langle (1-\chi)\psi_j, \psi_j\rangle = (2\pi h)^d \mathrm{Tr}\, \Pi_{[a,b]}(1-\chi) + o(1).$$

□

The following lemma allows us to take arbitrary collections of orthogonal quasimodes and complete them. Hence, we only have to prove results for complete sets of quasimodes and they follow for positive density sets.

Lemma 4. *Let $\{(\psi_j, E_j), j = 1, ..., J\}$ be a set of almost orthogonal quasimodes with*

$$\|\psi_j\|_{L^2} = 1, \quad \|(P - E_j)\psi_j\|_{L^2} = o(h^{2d+1}), \quad |\langle \psi_j, \psi_k \rangle| = o(h^{2d}), \ j \neq k.$$

Then, there exists $\{\varphi_k, k = 1, ...K\}$ such that the set $\{\varphi_k, k = 1, ..., K\} \cup \{\psi_j, j = 1, ..., J\}$ is a complete set of quasimodes for P.

Proof. Let u_j be eigenfunctions of P corresponding to E_j. Select an energy E. Then, by the proof of Lemma 1, there are gaps at distance $c_1(h)$ of size h^{d+1} in the spectrum of P near E. Let $\Lambda = \{u_k : |E_j - E| \leq c_1(h)\}$. Let $\Lambda' := \{\psi_j : |E_j - E| = o(h^{d+1})\}$. Then, for $\psi_j \in \Lambda'$, we have that

$$\psi_j = \sum_{u_k \in \Lambda} c_{jk} u_k.$$

Now, define $N := |\Lambda|$ and $M = |\Lambda'|$ letting $b_j = (c_{j1}, c_{j2}, ..., c_{jN})$, $j = 1, ..M$. Let $\{e_1, ..., e_m\}$ be an orthonormal basis for span $(\{b_j : j = 1, ..., M\})$. Apply the Gram-Schmidt process to obtain an orthonormal basis $\{e_1, ..., e_M, v_{M+1}, ..., v_N\}$ of $\mathbb{R}^N$ where

$$v_k = (v_{k1}, ..., v_{kN}).$$

Then, letting

$$\varphi_k = \sum_j v_{kj} u_j, \quad M + 1 \leq k \leq N,$$

$\{\psi_1, ..., \psi_M, \varphi_{M+1}, ..., \varphi_N\}$ is an almost orthonormal basis for span Λ. Repeating this process for each cluster, we obtain a complete set of quasimodes. □

We also need the following restatement of results in [3, Section 4.3] that is found in [5, Lemma A1].

Lemma 5. *Fix $T > 0$. Assume that $A \in \Psi^{comp}(M^o)$ is supported away from the boundary of M and $WF_h(A) \subset p^{-1}([a, b]) \setminus \mathcal{B}_T$. Then, for each $\chi \in C_c^\infty(M^o)$ and for each $t \in [-T, T]$, the operator $\chi e^{-itP/h} A$ is a Fourier integral operator supposted away from ∂M and associated to the restriction of φ_t to a neighborhood of $WF_h(A) \cap \varphi_t^{-1}(supp\ \chi)$, plus an $O_{L^2 \to L^2}(h^\infty)$ remainder. The following version of Egorov's Theorem holds:*

$$\chi e^{itP/h} A e^{-itP/h} = A_{t,\chi} + O(h^\infty)_{L^2(M) \to L^2(M)},$$

where $A_{T,\chi} \in \Psi^{comp}(M^o)$ is supported away from ∂M and $\sigma_h(A_{T,\chi}) = \chi(a \circ \varphi_t)$.

Now, we prove Theorem 1, following [5, Appendix A]

Proof. Let (ψ_j, E_j) be a positive density set of quasimodes. By Lemma 4, and the fact that our original quasimodes have positive density, we may assume without loss that (ψ_j, E_j) is a complete set of quasimodes (see definition 2).

We first show that

$$\limsup_{h\to 0} h^d \sum_{E_j\in[a,b]} \left| \langle B\psi_j, \psi_j\rangle - \fint_{p^{-1}(E_j)} \sigma_h(B) d\mu_{E_j} \right| = 0.$$

Then, by a standard diagonal argument that can be found, for example, in [17, Theorem 15.5], we extract the set $\Lambda(h)$.

Take a', b' such that $a' < a < b < b'$ and the assumptions (1.2) and (1.1) hold for $E \in [a', b']$. (If they do not hold when $E \notin [a, b]$, then we need to take a', b' getting close to a and b. e.g. Let $a' = a - 1/T$ and $b' = b + 1/T$. Then estimate the extra contributions by the Weyl law.) Now, fix a $T > 0$ and choose $\chi_T \in C_c^\infty(M^o)$ with $0 \leq \chi_T \leq 1$ and

$$\int_{T^*M\cap p^{-1}([a'-1,b'-1])} (1-\chi_T)^2 d\mu_\sigma \leq T^{-1}.$$

Let $\psi \in C_c^\infty(a'-1, b'+1)$ have

$$\psi(E)\int_{p^{-1}(E)} \chi_T d\mu_E = \int_{p^{-1}(E)} \sigma_h(B) d\mu_E,\ E \in [a', b'].$$

Then, by Lemma 3, it is enough to show that the conclusion holds for $B - \psi(P(h))\chi_T$, whose symbol integrates to 0 on $p^{-1}(E)$. Therefore, we assume, without loss, that

$$\int_{p^{-1}(E)} \sigma_h(B) d\mu_E = 0,\ E \in [a', b'].$$

By elliptic estimates, we may assume that $WF_h(B) \subset p^{-1}((a', b'))$. More specifically, $B \in \Psi^{comp}$. Thus, we can write $B = B'_T + B''_T$, where $WF_h(B'_T) \cap \mathcal{B}_T = \emptyset$ and $\|\sigma_h(B''_T)\|_{L^2(p^{-1}[a',b'])} \leq T^{-1}$.

Then, by Hölder's inequality, Lemma 2 and [5, Lemma 2.2], we have that

$$\begin{aligned} h^d \sum_{E_j\in[a,b]} |\langle B''_T\psi_j, \psi_j\rangle| \leq C\left(h^d \sum_{E_j\in[a,b]} \|B''_T\psi_j\|^2\right)^{1/2} &= C\left(h^d \sum_{E_j\in[a,b]} \|B''_T u_j\|^2\right)^{1/2} + o(1) \\ &\leq C\|\sigma_h(B''_T)\|_{L^2(p^{-1}[a',b'])} + o(1). \end{aligned}$$

Hence, the contribution of B''_T goes to 0 in the limit $\lim_{T\to\infty}\limsup_{h\to 0}$ and we may replace B by B'_T.

Now, by Duhamel's formula and the unitarity of $e^{itP/h}$,

$$e^{itP/h}\psi_j = e^{itE_j/h}\psi_j + \frac{i}{h}\int_0^t e^{i(t-s)P/h} o(h^{2d+1}) = e^{itE_j/h}\psi_j + o_T(h^{2d}).$$

So, defining

$$\langle A\rangle_T := \frac{1}{T}\int_0^T e^{itP/h} A e^{-itP/h} dt,$$

and using Lemma 2 and Hölder's inequality, we have

$$\begin{aligned} h^d \sum_{E_j \in [a,b]} |\langle B'_T \psi_j, \psi_j \rangle| &= h^d \sum_{E_j \in [a,b]} |\langle B'_T e^{-itE_j/h} \psi_j, e^{-itE_j/h} \psi_j \rangle| = h^d \sum_{E_j \in [a,b]} |\langle \langle B'_T \rangle_T \psi_j, \psi_j \rangle| + o_T(h^{2d}) \\ &\leq \left(h^d \sum_{E_j \in [a,b]} (\|\langle B'_T \rangle_T \psi_j\| + o_T(h^{2d}))^2 \right)^{1/2} = \left(h^d \sum_{E_j \in [a,b]} \|\langle B'_T \rangle_T u_j\|^2 \right)^{1/2} + o_T(1) \end{aligned}$$

From this point forward, the proof is identical to that in [5, Theorem A.2]. By Lemma 5, $\langle B'_T \rangle_T \chi_T$ is, up to an $O(h^\infty)_{L^2 \to L^2}$ remainder, a pseudodifferential operator in Ψ^{comp} compactly supported inside M^o and with principal symbol

$$\sigma_h(\langle B'_T \rangle_T \chi_T) = \frac{\chi_T}{T} \int_0^T \sigma_h(B'_T) \circ \varphi_t dt.$$

Now, all that remains to show is

$$\lim_{T \to \infty} \limsup_{h \to 0} h^d \sum_{E_j \in [a,b]} \|\langle B'_T \rangle_T u_j\|^2 = 0.$$

Since, by Lemma 3,

$$\limsup_{h \to 0} (2\pi h)^d \sum_{E_j \in [a,b]} \|(1 - \chi_T) u_j\|^2_{L^2} \leq T^{-1},$$

we can replace $\langle B'_T \rangle_T$ by $\langle B'_T \rangle \chi_T$. Thus, by [5, Lemma 2.2], it remains to show that

$$\lim_{T \to \infty} \|\sigma_h(\langle B'_T \rangle_T \chi_T)\|_{L^2(p^{-1}([a',b']))} \leq \lim_{T \to \infty} \|\langle \sigma_h(B'_T) \rangle_T\|_{L^2(p^{-1}([a',b']))} = 0. \tag{2.3}$$

To do this, write

$$\|\langle \sigma_h(B'_T) \rangle_T\|_{L^2(p^{-1}(E))} \leq \|\langle \sigma_h(B) \rangle_T\|_{L^2(p^{-1}(E))} + \|\langle \sigma_h(B''_T) \rangle_T\|_{L^2(p^{-1}(E))}.$$

The first term on the right goes to 0 when $T \to \infty$ by the von Neumann ergodic theorem and the second term is bounded by $\|\sigma_h(B''_T)\|_{L^2(p^{-1}(E))}$ and hence also goes to 0. □

Remark: Notice that (2.3) is the only step in which the ergodicity of the flow is used. This will be important when we adapt the result to ergodic invariant subsets of phase space.

3. From Semiclassical to Classical Quantum Ergodicity

For completeness and to present the proof in the quasimode case, we will now pass from Theorem 1 with $P = -h^2\Delta$ to Theorem 2.

Proof. Let $h^2\lambda_j^2 = E_j$ and $\lambda = h^{-1}$, $\chi \in C^\infty$ $\chi(\xi) \equiv 0$ for $|\xi| \leq 1$ and $\chi \equiv 1$ for $|\xi| \geq 2$, and $\chi_\epsilon = \chi(\xi/\epsilon)$.

Let $\hat{A}$ be a classical pseudodifferential operator of order 0 on M with symbol compactly supported away from ∂M. Define $a_0 := \sigma(\hat{A})$. Then, a_0 is homogeneous degree 0 on T^*M. Hence $a_0(x, D)\chi(D) = a_0(x, hD)\chi(hD/h)$. Now, define $A_\epsilon \in \Psi^0(M^o)$ by

$$A_\epsilon := a(x, hD)\chi_\epsilon(hD).$$

Theorem 1 gives, for $0 < a < b$,

$$h^d \sum_{h\lambda_j \in [a,b]} \left| \langle A_\epsilon \psi_j, \psi_j \rangle - \fint_{p^{-1}(E_j)} \sigma_h(A_\epsilon) d\mu_{E_j} \right| \to 0, \; h = \lambda^{-1} \to 0.$$

But, since $\sigma_h(A_\epsilon) = a_0(x,\xi)\chi_\epsilon(\xi)$ and a_0 is homogeneous degree 0, we have

$$\fint_{p^{-1}(E_j)} \sigma_h(A_\epsilon) d\mu_{E_j} = \fint_{S^*M} \sigma(\hat{A}) d\mu_L + O(\epsilon).$$

Hence, we need to show,

$$\lim_{\epsilon \to 0} \limsup_{h \to 0} h^d \sum_{h\lambda_j \in [a,b]} \langle (\hat{A} - A_\epsilon)\psi_j, \psi_j \rangle = 0.$$

By Hölder's inequality, Lemma 2, and [5, Lemma 2.2],

$$\begin{aligned} h^d \sum_{h\lambda_j \in [a,b]} |\langle (\hat{A} - A_\epsilon)\psi_j, \psi_j \rangle| &\leq C \left(h^d \sum_{h\lambda_j \in [a,b]} \|(\hat{A} - A_\epsilon)\psi_j\|^2 \right)^{1/2} = C \left(h^d \sum_{h\lambda_j \in [a,b]} \|(\hat{A} - A_\epsilon)u_j\|^2 \right)^{1/2} + o(1) \\ &\leq C \|\sigma_h(\hat{A} - A_\epsilon)\|_{L^2(p^{-1}([a,b]))} + o(1). \end{aligned}$$

But, $\lim_{\epsilon \to 0} \|\sigma_h(\hat{A} - A_\epsilon)\|_{L^2(p^{-1}([a,b]))} = 0$.

Thus, for any $\delta > 0$, we have

$$\limsup_{h \to 0} h^d \sum_{h\lambda_j \in [\delta,1]} \left| \langle \hat{A}\psi_j, \psi_j \rangle - \fint_{S^*M} \sigma(\hat{A}) d\mu_L \right| = 0.$$

All that remains is to show that

$$\lim_{\delta \to 0} \limsup_{h \to 0} h^d \sum_{h\lambda_j \in [0,\delta)} \left| \langle \hat{A}\psi_j, \psi_j \rangle - \fint_{S^*M} \sigma(\hat{A}) d\mu_L \right| = 0.$$

But, letting $\bar{a}_0 = \fint_{S^*M} \sigma(\hat{A}) d\mu_L$, and applying the Weyl law, and Hölder's inequality,

$$h^d \sum_{h\lambda_j \in [0,\delta)} |\langle \hat{A}\psi_j, \psi_j \rangle - \bar{a}_0| \leq C \left(h^d \sum_{h\lambda_j \in [0,\delta)} \|(\hat{A} - \bar{a}_0)\psi_j\|^2 \right)^{1/2} = O(\delta^d) \|(\hat{A} - \bar{a}_0)\|_{L^2 \to L^2} + O_\delta(h).$$

Once again, to obtain the full density subsequence, we employ a standard diagonal argument. □

4. Quantum Ergodicity for Subsets

Now, we prove Theorem 3 using the fact that only step (2.3) uses the ergodicity of the flow. For completeness, we include the following elementary lemma.

Lemma 6. *Let U satisfy* (1.4), *let μ_2 be a finite measure on S^*M and suppose that for all $a \in C^\infty(S^*M)$ compactly supported away from ∂M with $\int_U a d\mu_L = 0$, we have $\int_U a d\mu_2 = 0$. Then, $\mu_2|_U \equiv c\mu_L|_U$ for some $c \geq 0$.*

Proof. Let $\mu_1 = (\mu_L(U))^{-1}\mu_L$. Then, $\mu_1(U) = 1$. Define $\chi \in C^\infty(S^*M)$ compactly supported away from ∂M with $0 \le \chi \le 2$ and $\int_U \chi d\mu_1 = 1$ and $\int_U \chi d\mu_2 = c_2 > 0$. (To see that such functions exist simply take non-negative approximations to 1_U with support inside M^o.)

Now, let $a \in C^\infty(S^*M)$ compactly supported away from ∂M with $\int_U a d\mu_1 = \bar{a}$. Then, $\int_U a - \bar{a}\chi d\mu_1 = 0$ and hence $\int_U a - \bar{a}\chi d\mu_2 = 0$. Therefore, for $a \in C^\infty(S^*M)$ compactly supported away from ∂M,

$$\int_U a d\mu_2 = c_2 \int_U a d\mu_1.$$

But, $\mu_1|_U$ and $\mu_2|_U$ are positive distributions of order 0 since U satisfies (1.4). Hence, since

$$\int_U a d\mu_2 = \int_U a d(c_2\mu_1) = 0$$

for all $a \in C^\infty(S^*M)$ compactly supported away from ∂M, and M can be exhausted by compact subsets, $\mu_2|_U \equiv c_2\mu_1|_U$. □

We now prove Theorem 3.

Proof. Lemma 4 shows that without loss of generality we may work with ψ_j forming a complete set of quasimodes. Fix $a \in C^\infty(S^*M)$ compactly supported away from ∂M with

$$\int_U a d\mu_L = 0. \tag{4.1}$$

Fix $\epsilon > 0$. Then, let U_ϵ be open with $\overline{U} \subset U_\epsilon$ and $\mu_L(U_\epsilon \setminus U) < \epsilon$ (note that we use (1.4) here). Let $\chi_\epsilon \in C_c^\infty(U_\epsilon)$ compactly supported away from ∂M with $\chi_\epsilon|_{\bar{U}} \equiv 1$, $0 \le \chi_\epsilon \le 1$. Then, let $a_\epsilon = \chi_\epsilon a$. Based on the arguments used to prove Theorems 1 and 2, we have that

$$\limsup_{\lambda\to\infty} \lambda^{-d} \sum_{\lambda_j \le \lambda} \left|\langle A_\epsilon \psi_j, \psi_j\rangle\right| \le C\|\langle a_\epsilon\rangle_T\|_{L^2(S^*M)}, \tag{4.2}$$

where $A_\epsilon = a_\epsilon(x, D)$,

$$\langle a_\epsilon\rangle_T := \frac{1}{T}\int_0^T a_\epsilon \circ \varphi_t(x,\xi)dt.$$

We have that φ_t is ergodic on U and U is invariant under φ_t. Hence, by the von Neumann ergodic theorem $\langle a_\epsilon 1_U\rangle_T \to \fint_U a_\epsilon d\mu_L = 0$ in L^2. Again, by the ergodic theorem, $\langle a_\epsilon 1_{U^c}\rangle_T \to Pa_\epsilon$ in L^2 with $\|Pa_\epsilon\|_{L^2} \le C\|a\|_{L^\infty}\epsilon$. Hence,

$$\lim_{T\to\infty} \|\langle a_\epsilon\rangle_T\|_{L^2(S^*M)} \le C\|a\|_{L^\infty}\epsilon. \tag{4.3}$$

We now show that one can extract a full density subsequence ψ_{j_k} such that for all $a \in C^\infty(S^*M)$ satisfying (4.1)

$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\langle A_\epsilon \psi_{j_k}, \psi_{j_k}\rangle = 0.$$

To do this, fix $\epsilon > 0$ and let

$$\Gamma(\epsilon) = \{\lambda_j \le \epsilon^{-1} : |\langle A_\epsilon\psi_j, \psi_j\rangle| \ge (C\|a\|_{L^\infty}\epsilon)^{\frac{1}{2}}\}.$$

Then, by the Chebyshev inequality, (4.2), and (4.3),

$$\epsilon^d|\Gamma(\epsilon)| \leq (C\epsilon\|a\|_{L^\infty})^{\frac{1}{2}}$$

and for $\lambda_j \notin \Gamma(\epsilon)$

$$|\langle A_\epsilon\psi_j, \psi_j\rangle| \leq (C\epsilon\|a\|_{L^\infty})^{\frac{1}{2}}.$$

But, by the Weyl law,

$$|\Gamma(\epsilon)|/|\{\lambda_j \leq \epsilon^{-1}\}| = \epsilon^d|\Gamma(\epsilon)|/\left(\mathrm{Vol}(B^*M) + o(1)\right) \leq C(\|a\|_{L^\infty}\epsilon)^{1/2}.$$

Therefore, $\Lambda(\epsilon) = \{\lambda_j \leq \epsilon^{-1}\} \setminus \Gamma(\epsilon)$ has full density as $\epsilon \to 0$. Now, we take a countable dense set, $\{a_k\}$ of $C^\infty(S^*M) \cap \{a : \|a\|_{L^\infty} \leq 1,\ a \text{ satisfies (4.1)}\}$ and apply a variant of the standard diagonal argument (contained for example, in [17, Theorem 15.5]).

Now, suppose that ψ_{j_k} is a further subsequence with defect measure μ. Then, for $a \in C^\infty(S^*M)$ satisfying (4.1), we have that

$$\lim_{k\to\infty}\langle A_\epsilon\psi_{j_k}, \psi_{j_k}\rangle = \int a_\epsilon d\mu = o(1).$$

Hence, by the dominated convergence theorem,

$$\int_U a d\mu = 0.$$

Thus, we may apply Lemma 6 to obtain the result. □

Remark: We may also make similar arguments to obtain a result in the semiclassical setting.

5. An example

We now present an example to which Theorem 3 applies. Let Ω be a symmetric mushroom billiard, as in [2], composed of a hat that is a semicircle of radius 1 and a base that has width w and height h (figure 1 shows such a mushroom with two billiard flows). Then $S^*\Omega$ has a subset U satisfying (1.4) and (1.5) (see [2]). Let

$$u_{nk} = \sin(n\theta)J_n(\alpha_{n,k}r)$$

be the eigenfunctions of the Dirichlet Laplacian on the semidisk. Here J_n denotes the Bessel function of the first kind of order n and $\alpha_{n,k}$ the k^{th} positive zero of J_n.

From [10, Appendix A], we have that for $0 < \gamma < \frac{2}{3}$, $z \in (0, 1 - n^{\gamma-\frac{2}{3}})$

$$0 < J_n(nz) < 2^{-n^{\gamma/3}}.$$

Also, from [11], we have for $0 \leq k < c_2 n$, that $n < \alpha_{n,k} < Cn$ for some $C > 0$. Thus, $J_n(\alpha_{n,k}r) = O(2^{-n^{\gamma/3}})$ for $r < \frac{1}{2C}$.

Now, suppose $0 < w < \frac{1}{4C}$. Then, let $\chi \in C^\infty(\Omega)$ with $\chi \equiv 0$ in $|r| < \frac{1}{4C}$ and $\chi \equiv 1$ in $|r| \geq \frac{1}{2C}$. If we let

$$v_{n,k} := \chi u_{n,k},$$

for $0 \leq k < c_2 n$, and extend by 0 outside of the hat of Ω, then $v_{n,k}$ are quasimodes for the Dirichlet Laplacian on Ω with

$$(-\Delta - \alpha_{n,k}^2)v_{n,k} = O(2^{-n^{\gamma/3}}).$$

Now, by the orthogonality of $u_{n,k}$, $v_{n,k}$ are orthogonal up to $O(e^{-Cn^{\gamma/3}})$. Hence, $\{v_{n,k}\}$ are a family of $\frac{c_2 n(c_2 n+1)}{2}$ almost orthogonal quasimodes with $O(e^{-Cn^{\gamma/3}})$ error. Thus,

$$\{v_{n,k}, \alpha_{n,k}^2, 0 \leq k < c_2 n, n = 1, ...\}$$

form a positive density set of quasimodes for $-\Delta$ on $[0, 1]$.

Since the $v_{n,k}$ are $O(n^{-\infty})$ quasimodes, $WF_h(v_{n,k})$ is invariant under the Hamiltonian flow (see, for example, [17, Section 12.3]). Therefore, since $WF_h(v_{n,k})$ does not intersect the foot of Ω, the quasimodes must concentrate away from the ergodic set, U. Thus, $v_{n,k}|_U$ has a defect measure μ. Hence, Theorem 3 applies and the constant we obtain is 0.

References

[1] A. H. Barnett and T. Betcke. Quantum mushroom billiards. Chaos 17 (2007), no. 4, 043125, 13 pp.

[2] L. A. Bunimovich. Mushrooms and other billiards with divided phase space Chaos. Volume 11 number 4 (2001), 802-808.

[3] H. Christianson. Quantum monodromy and non-concentration near a closed semi-hyperbolic orbit. Trans. Amer. Math. Soc. 363(2011), no. 7 3373-3438.

[4] Y. Colin de Verdiére. Ergodicité et fonctions propres du laplacien. Commun. Math. Phys. 102, (1985) 497-502.

[5] S. Dyatlov and M. Zworski. Quantum ergodicity for restrictions to hypersurfaces. arXiv:1204.0284v2.

[6] L. Hörmander, *The Analysis of Linear Partical Differential Operators, Volume III*, Springer, 1985.

[7] V. F. Lazutkin. KAM Theory and Semiclassical Approximations to Eigenfunctions. Springer-Verlag Berlin Heidelberg, USA, 1993.

[8] J. Marklof and S. O'Keefe. Weyl's law and quantum ergodicity for maps with divided phase space. Nonlinearity 18 (2005), no. 1, 277−304.

[9] J. Marklof and Z. Rudnick, Almost all eigenfunctions of a rational polygon are uniformly distributed. J. Spectr. Theory 2 (2012), 107−113.

[10] B.T. Nguyen and D. S. Grebenkov. Localization of Laplacian eigenfunctions in circular, spherical and elliptical domains. arXiv:1203.5022v1 22 Mar 2012.

[11] F. W. J. Olver, The asymptotic expansion of Bessel functions of large order. Phil. Trans. of the Royal Society of London. Series A, Mathematical and Physical Sciences. Vol. 247, No. 930 (1954), 328-368.

[12] G. Riviére. Remarks on Quantum Ergodicity. arXiv:1209.3576.

[13] I. C. Percival. J. Phys. B pp. L229232 (1973).

[14] A. Shnirelmann. Ergodic properties of eigenfuncions. Usp. Math. Nauk. 29, (1974) 181−182.

[15] S. Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. Duke Math. J. 55, (1987) 919-941.

[16] S. Zelditch and M. Zworski. Ergodicity of eigenfunctions for ergodic billiards. Comm. Math. Phys. 175 (1996), 673−682.

[17] M. Zworski. *Semiclassical Analysis*, Graduate Studies in Mathematics, AMS, 2012.

Mathematics Department, University of California, Berkeley, CA 94720, USA

E-mail address: jeffrey.galkowski@math.berkeley.edu